# Chapter 11: Infinite Sequences and Series

## Section 1:

**Definition 1:**
A numerical sequence, or simply a sequence is a function a, N → R.

**Theorem 1:**
$\{a_n\}$ is a sequence. We say that $\{a_n\}$ converges to a limit L as n → ∞, and we write $\lim_{n\to\infty} a_n = L$, if to each given $\varepsilon > 0$, there corresponds a positive integer N such that $n \geq N \Rightarrow |a_n - L| < \varepsilon$

**Theorem 2:**
$\lim_{n\to\infty} k = k$ (Where k is a function).

$\lim_{n\to\infty} \frac{1}{n} = 0$

**Definition 2:**
$\{a_n\}$ is a sequence. If $a_n$ does not converge, we say that $\{a_n\}$ diverge.

**Theorem 3:**
$\{a_n\}$ and $\{b_n\}$ are sequence. If $a_n \to A$ and $b_n \to B$ the

i. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$
ii. $\lim_{n\to\infty} a_n b_n = AB$
iii. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ provided $b_n$ is nonzero and $B$ is also nonzero.

**Theorem 4: The Sandwich Theorem:**
$\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences

$$\left.\begin{array}{l} a_n \leq b_n \leq c_n \quad \forall n \\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \end{array}\right\} \Rightarrow \lim_{n\to\infty} b_n = L$$

**Corollary to Sandwich Theorem (SCT):**
$\{a_n\}$, $\{c_n\}$ are sequences:

$$\left.\begin{array}{l} |a_n| \leq c_n \quad \forall n \\ c_n \to 0 \end{array}\right\} \Rightarrow a_n \to 0$$

**Theorem 5:**
$\{a_n\}$ is a sequence, $f(x)$ is a function:

$$\left.\begin{array}{l} a_n \to L \\ f(x) \text{ continuous at L} \end{array}\right\} \Rightarrow f(a_n) \to f(L)$$

**Theorem 6:**
$\{a_n\}$ is a sequence, $f(x)$ is a function:

$$\left.\begin{array}{l} a_n = f(n) \quad \forall n \\ \lim_{n\to\infty} f(x) = L \end{array}\right\} \Rightarrow a_n \to L$$

**Theorem 7: Six important limits:**

$$\lim_{n\to\infty}\frac{\ln n}{n^{\alpha}}=0 \quad \text{with } \alpha>0$$

$$\lim_{n\to\infty}\sqrt[n]{n}=1$$

$$\lim_{n\to\infty}x^{\frac{1}{n}}=1 \quad \text{with } x>0$$

$$\lim_{n\to\infty}x^{n}=0 \quad \text{with } |x|<1$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=e^{x} \quad \text{with } x\in IR$$

$$\lim_{n\to\infty}\frac{x^{n}}{n!}=0 \quad \text{with } x\in IR$$

**Definition 3:**

$\{a_n\}$ is a sequence. $M \in IR$.

i. We say that $a_n$ is decreasing if $a_{n+1}\le a_n \quad \forall n$

ii. We say that $a_n$ is bounded from above by $M$ if $a_n \le M \quad \forall n$

**Theorem 8: Important Theorem (IT):**

$$\left.\begin{array}{l}\{a_n\}\text{ increasing}\\ \{a_n\}\text{ bounded from above}\end{array}\right\}\Rightarrow \{a_n\}\text{ converges.}$$

$$\left.\begin{array}{l}\{a_n\}\text{ decreasing}\\ \{a_n\}\text{ bounded from below}\end{array}\right\}\Rightarrow \{a_n\}\text{ converges}$$

$\{a_n\}$ unbounded $\Rightarrow \{a_n\}$ diverges.

## Section 2:

**Definition 1:**

$\{a_n\}$ is a sequence. The expression $\sum_{n=1}^{\infty}a_n=a_1+a_2+a_3+....$ is called an infinite series.

The sequence of partial sum of $\sum_{n=1}^{\infty}a_n$ is the sequence $\{S_n\}$ defined by: $S_1=a_1$ , $S_2=a_1+a_2$

$S_n=a_1+a_2+a_2+....+a_n$.

We have: $\sum_{n=1}^{\infty}a_n\begin{cases}=\lim\limits_{n\to\infty}S_n & \text{if } \{S_n\} \text{ converges}\\ \text{diverges if} & \{S_n\} \text{ diverges}\end{cases}$

**Geometric series:**

$a,r\in IR,\ a\neq 0$

$\sum_{n=1}^{\infty}ar^{n-1}=a+ar+ar^2+...$ is called a geometric series of ratio $r$.

The sequence of partial sum $S_n=\begin{cases}a\dfrac{1-r^n}{1-r} & \text{if } r\neq 1\\ na \text{ if } r=1\end{cases}$ and $\sum_{n=1}^{\infty}ar^{n-1}=\begin{cases}\dfrac{a}{1-r} & \text{if } |r|<1\\ \text{diverges if } |x|\ge 1\end{cases}$

**Theorem 1:**

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series and $k \in IR$. If $\sum_{n=1}^{\infty} a_n = \text{A}$ and $\sum_{n=1}^{\infty} b_n = B$ then:

i. $\sum_{n=1}^{\infty} a_n \pm b_n = A \pm B$

ii. $\sum_{n=1}^{\infty} ka_n = kA$

**Theorem 2: The n^th term test:**

$\sum_{n=1}^{\infty} a_n$ is a series. If $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow a_n \to 0$

Equivalently: $a_n \not\to 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(The converse of this theorem is not always true).

**Useful Remark: (UR):**

$\{a_n\}$ is a sequence. $a_n \to 0 \Rightarrow |a_n| \to 0$. Equivalently, $|a_n| \not\to 0 \Rightarrow a_n \not\to 0$.

**Important Remark: (IR):**

$\sum_{n=1}^{\infty} a_n$ is a series, $N$ is a positive integer. Then, $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=N}^{\infty} a_n$ converges.

# Section 3:

**Definition 1:**

If $a_n \geq 0 \;\; \forall n$, we say $\sum_{n=1}^{\infty} a_n$ is a series of nonnegative terms.

If $a_n > 0 \;\; \forall n$, we say $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

**Corollary to IT: (CIT):**

$\sum_{n=1}^{\infty} a_n$ is a series of nonnegative terms. $S_n$ is the series of partial sum of $\sum_{n=1}^{\infty} a_n$.

Then: $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \{S_n\}$ is bounded from above.

**Theorem 1: Integral test:**

$\sum_{n=1}^{\infty} a_n$ is a series of nonnegative terms. $f(x)$ is a function that is continuous and decreasing on $(0, \infty)$ with $a_n = f(n) \;\; \forall n$. Then: $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \int_1^{\infty} f(x)dx = \lim_{b \to \infty} \int_1^{b} f(x)dx$ converges.

## Section 4:

**Theorem 1: Direct Comparison Test: (DCT):**

Given $\sum_{n=1}^{\infty} a_n$ , $\sum_{n=1}^{\infty} c_n$ , and $\sum_{n=1}^{\infty} d_n$ series of nonnegative terms.

Then:

i. $a_n \le c_n \quad \forall n :$ $\sum_{n=1}^{\infty} c_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

ii. $a_n \ge d_n \quad \forall n :$ $\sum_{n=1}^{\infty} d_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

**Theorem 2: Limit Comparison Test: (LCT):**

$\sum_{n=1}^{\infty} a_n$ series of nonnegative terms.

$\sum_{n=1}^{\infty} b_n$ series of positive terms.

i. $$\left.\begin{array}{l} \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L \\ 0 < L < \infty \end{array}\right\} \Rightarrow \sum_{n=1}^{\aleph} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ both converge of both diverge.}$$

ii. $$\left.\begin{array}{l} \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0 \\ \sum\limits_{n=0}^{\infty} b_n \text{ converges} \end{array}\right\} \Rightarrow \sum_{n=1}^{\aleph} a_n \text{ converges}$$

iii. $$\left.\begin{array}{l} \lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty \\ \sum\limits_{n=1}^{\infty} b_n \text{ diverges} \end{array}\right\} \Rightarrow \sum_{n=1}^{\aleph} a_n \text{ diverges}$$

## Section 5:

**Theorem 1: The Ratio Test:**

Suppose $\sum_{n=1}^{\infty} a_n$ is a series of positive terms.

Suppose $\rho = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$, then:

i. $\rho < 1 \Rightarrow$ series converges

ii. $\rho > 1 \Rightarrow$ series diverges

iii. $\rho = 1 \Rightarrow$ Test fails

**Theorem 2: The Root Test:**

Suppose $\sum_{n=1}^{\infty} a_n$ is a series of nonnegative terms.

Suppose $\rho = \lim\limits_{n\to\infty} \sqrt[n]{a_n}$ , then:

iv. $\rho < 1 \Rightarrow$ series converges

v. $\rho > 1 \Rightarrow$ series diverges

vi. $\rho = 1 \Rightarrow$ Test fails

## Section 6:

**Definition 1:**

Suppose $\{a_n\}$ is a sequence of positive terms, then: $\sum_{n=1}^{\infty}(-1)^n a_n = a_1 - a_2 + a_3 - a_4 + \dots$ is called an alternating series.

**Theorem 1: The Alternating Series Test (AST):**

$$\text{Suppose } \left.\begin{array}{l} a_n > 0 \ \forall n \\ a_{n+1} \le a_n \ \ \forall n \\ a_n \to 0 \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty}(-1)^n a_n \text{ converges.}$$

**Theorem 2: Alternating Series Estimation Theorem (ASET):**

$\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ is an alternating series. $\{S_n\}$ is the sequence of partial sums. Then:

$\left.\begin{array}{l} a_n > 0 \ \forall n \\ a_{n+1} \le a_n \ \ \forall n \\ a_n \to 0 \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty}(-1)^n a_n = S_n + error$, with $|error| \le |\text{first unused term}|$, and "*error*" has the same sign as the first unused term.

**Theorem 3: Absolute Convergence Test (ACT):**

$\sum_{n=1}^{\infty} a_n$ is a series: $\sum_{n=1}^{\infty}|a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$.

**Definition 2:**

$\sum_{n=1}^{\infty} a_n$ is a series that converges. If $\sum_{n=1}^{\infty}|a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

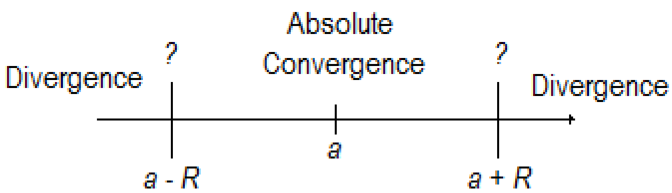
## Section 7:

**Definition 1:**

A series of the form $\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$ is called a power series of center a, and of coefficient $c_0, c_1, c_2, \dots$

**Theorem 1:**

$\sum_{n=1}^{\infty} c_n (x-a)^n$ is a power series, then:




**Theorem 2:**

$\sum_{n=1}^{\infty} c_n (x-a)^n$ is a power series with radius of convergence $R$. Suppose $I = (a-R; a+R)$ the internal of interval of convergence.

Suppose $f(x)=\sum_{n=1}^{\infty} c_n (x-a)^n$ with $x \in I$ , then this function is infinitely differentiable on *I*, and

$$f'(x)=\sum_{n=1}^{\infty} nc_n (x-a)^{n-1} \quad \text{with } x \in I$$

$$f''(x)=\sum_{n=1}^{\infty} n(n-1)c_n (x-a)^{n-2} \quad \text{with } x \in I$$

**Theorem 3:**

$\sum_{n=1}^{\infty} c_n (x-a)^n$ is a power series with radius of convergence *R* and suppose $I=(a-R;a+R)$ the internal of interval of convergence, then:

$\int f(x)dx$ exists for all *x* in *I*, and $\int f(x)dx = c + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ for $x \in I$

**Theorem 4:**

$\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are 2 power series with radii of convergence $R_1$ and $R_2$ respectively.

Suppose *R* is the minimum of $R_1$ and $R_2$, then: $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)=\sum_{n=0}^{\infty} c_n x^n$ for $|x|<R$

where $c_n = \sum_{k=0}^{n} a_{n-k} b_k$ .

# Section 8:

**Definition 1:**

I is a open interval of center *a*.
*f*(*x*) is a function which is infinitely differentiable on *I*.
n a positive integer.
Then: The Taylor polynomial of order n generated by *f* at $x=a$ is:

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(x)}{2!}(x-a)^2+........+\frac{f^{(n)(a)}}{n!}(x-a)^n$$

The Taylor series generated by $f(x)$ at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(x)}{2!}(x-a)^2+........$$

# Section 9:

**Taylor's Theorem:**

*I* is an open interval of center *a*.
*f*(*x*) is a function which is infinitely differentiable on *I*.
Suppose $P_n(x)$ is a Taylor polynomial of order n generated by *f* at $x=a$.

Then: $f(x)=P_n(x)+R_n(x)$ with $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some *c* strictly between 0 and *x*.

# Section 10:

**Theorem 1: The Binomial Theorem:**

$m \in IR$, $(1+x)^m = 1 + m + \sum_{n=2}^{\infty} \binom{m}{n} x^n$ for $|x| < 1$ where $\binom{m}{n} = \frac{m(m-1)(m-2)......[m-(m-1)]}{n!}$

# Section 11:

**Definition 1:**

*I* = [*a*, *b*] is a closed interval, and $f(x)$ function defined on *I*.

We say that *f* is piecewise continuous on *I* if:

i. $f$ has only finitely many points of discontinuity on .

ii. If $x_0 \in (a,b)$ is a point of discontinuity of *f*, then $f(x_0^-) = \lim_{x \to x_0^-} f(x)$ and $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ both exists.

iii. $f(a^+) = \lim_{x \to a^+} f(x)$ and $f(a^-) = \lim_{x \to b^-} f(x)$ both exists.

**Definition 2:**

$f(x)$ is a piecewise continuous function on $[0,2\pi]$. The Fourier series of $f$ is defined as

$a_0 + 2\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ with:

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos nx \, dx \qquad n = 1, 2, 3, ......$$

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin nx \, dx \qquad n = 1, 2, 3, ......$$

**Theorem 1:**

$f(x)$ is a function defined on $[0,2\pi]$

if *f* and *f′* are piecewise continuous on $[0,2\pi]$, then : continuity discontinuity

$$a_0 + 2\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \begin{cases} f(x) & \text{if } x \text{ is a point of continuity of } f \\ \frac{f(x^-) + f(x^+)}{2} & \text{if } x \text{ is not a point of continuity of } f \\ \frac{f(0^+) + f(2\pi^-)}{2} & \text{if } x = 0 \\ \frac{f(2\pi^-) + f(0^+)}{2} & \text{if } x = 2\pi \end{cases}$$